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# The Compatibility between the Higher Dimensions Self Duality and the Yang-Mills Equation of Motion

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**Abstract**

We study the compatiblity between the higher dimension dualities and the Yang-Mills equation of motion. Taking a 't Hooft solution as a starting point, we come to the conclusion that for only 4 dimensions the self duality implies the equation of motion for generic instanton size. Whereas in higher dimensions, the self duality is compatable with the equation of motion, approximately, for small instanton size i.e. the zero curvature condition. At the mathematical level, the self duality is still useful since it transforms a second order into a first order differential equation.

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Duality play an important role in physics. It represent weak form of integrablity. In 4 dimensions, the Yang-Mills self dual solution, instanton[1], is a non-perturbative notion. Extending it to higher dimensions [2, 3, 4, 5, 6] is very important for studying p-branes solitons[7, 8, 9, 10, 11, 12] In recent article, we proposed a new formulation of the 8 dimensional octonionic instanton [13] which mimicks many of the quaternionic 4 dimensional instanton. Our starting point was the construction of a Clifford algebra, Cliff(0,7), from octonions

$$(E_i)_{\alpha\beta} = \delta_{i\alpha}\delta_{\beta 0} - \delta_{i\beta}\delta_{\alpha 0} + f_{i\alpha\beta}, \tag{1}$$

$f_{ijk}$ is the octonionic structure constant. Noticing its similarity with the 't Hooft matrices. Writing their commutation relations as

$$\begin{aligned} [E_i, E_j] &= 2f_{ijk}E_k - 2[E_i, 1|E_j], & (2)\\ &= 2(f_{ijk}1_{8\times 8} + [E_i, 1|E_j]E_k)E_k, & (3)\\ &= 2\rho_{ijk}E_k. & (4) \end{aligned}$$

So our structure constants become matrices[2] and our varying torsion over $S^7$ is

$$T(X, Y) = -2\rho_{ijk}. \tag{5}$$

Now, introducing

$$E_\mu \equiv (E_0 = 1, E_i) \quad ; \quad \bar{E}_\mu \equiv (E_0, -E_i), \tag{6}$$

we may define the following tensor

$$\vartheta_{\mu\nu} = \frac{1}{2}(\bar{E}_\mu E_\nu - \bar{E}_\nu E_\mu). \tag{7}$$

which is self dual with respect to

$$\eta_{0ijk} = \rho_{ijk} \quad \text{and zero elsewhere}, \tag{8}$$

explicitly, we have

$$\vartheta_{\alpha\beta} = \eta_{\alpha\beta\mu\nu}\vartheta_{\mu\nu} \tag{9}$$

Then our 't Hooft solution is

$$A_\mu(x) = \frac{x^2}{x^2+\lambda^2}g^{-1}(x)\partial_\mu g(x) = -\frac{\vartheta_{\mu\nu}x^\nu}{\lambda^2+x^2} \tag{10}$$

for

$$F_{\mu\nu} = \frac{\vartheta_{\mu\nu}2\lambda^2}{(\lambda^2+x^2)^2}, \tag{11}$$

and our self-duality is

$$F_{\alpha\beta} = \eta_{\alpha\beta\mu\nu}F_{\mu\nu} \tag{12}$$

[2] There is no summation in the second term of (3) to avoid an ugly 1/7 factor.

But the problem is in 4 dimensions, the self duality has a very deep geometrical meaning. It is equivalent to the Bianchi identities. So a self dual solution is automatically a solution of the equation of motion. Here, our solution does not satisfy the Yang-Mills equation of motion

$$D_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] \neq 0, \tag{13}$$

actually it comes too close

$$3\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] \neq 0. \tag{14}$$

In this letter, we would like to know the reason of this semilarity and if it possible to investigate the possibility of having higher dimensions duality that is equivalent to the equation of motion.

Rewriting a generic self dual $A_\mu$ as

$$A_\mu(x) = \frac{x^2}{(x^2+\lambda^2)^m} g^{-1}(x)\partial_\mu g(x) = a\frac{G_{\mu\nu}x^\nu}{(\lambda^2+x^2)^m} \tag{15}$$

for a generic self dual matrix - to be found - and imposing

$$F_{\mu\nu} = b\frac{G_{\mu\nu}}{(\lambda^2+x^2)^n}, \tag{16}$$

in order to get a self-duality as

$$F_{\alpha\beta} = \eta_{\alpha\beta\mu\nu}F_{\mu\nu}, \tag{17}$$

we find that

$$m = 1 \quad n = 2, \tag{18}$$

and

$$bG_{\mu\nu} = -2a(\lambda^2+x^2)G_{\mu\nu} + 2a(G_{\mu\alpha}x_\alpha x_\nu - G_{\nu\alpha}x_\alpha x_\mu) + a^2 x_\alpha x_\beta [G_{\mu\alpha}, G_{\nu\beta}], \tag{19}$$

leading to

$$b = -2\lambda^2 \tag{20}$$

and

$$(G_{\mu\alpha}\delta_{\nu\beta} - G_{\nu\alpha}\delta_{\mu\beta}) + \frac{a}{2}[G_{\mu\alpha}, G_{\nu\beta}] = \delta_{\alpha\beta}G_{\mu\nu}. \tag{21}$$

Whereas, taking $A_\mu$ and $F_{\mu\nu}$ and substituting in the YM equation of motion,

$$\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0, \tag{22}$$

we find

$$-4G_{\alpha\nu} + a[G_{\mu\alpha}, G_{\mu\nu}] = 0. \tag{23}$$

Now, we should find a common solution ($G_{\mu\nu}$) that satisfy both of (21) and (23). We can reformulate (21) for $\alpha = \beta$ then

$$(4 - 2dim)G_{\alpha\nu} + a[G_{\mu\alpha}, G_{\mu\nu}] = 0, \tag{24}$$

i.e (21) and (23) have a common 't Hooft-like solution if and only if

$$(4 - 2dim) = -4 \quad \Longrightarrow \quad dim = 4. \tag{25}$$

It is very easy to derive (14) directly from (25). From the mathematical point of view, we have acheived some success since we transformed a second order differential equation's problem (14) into the simple self duality relation (12) but physically, the self duality gives the YM equation of motion only for $\lambda \longrightarrow 0$ i.e. for very small instanton size. One can use some non-linear perturbation methods to construct other "acceptable" solutions.

Let's try to find another self dual tensor and a new self duality in 8 dimensions and see what happens. We can define a new tensor that plays the role of $\rho_{ijk}$ directly from octonions. We can use $f_{ijk}$ the octonionic structure constant and define

$$\pi_{0ijk} = f_{ijk} \quad \text{and zero elsewhere} \tag{26}$$

our self dual tensor may be

$$\sigma_{\mu\nu} = \vartheta_{\mu\nu} - [E_\mu, E_\nu], \tag{27}$$

leading to

$$\sigma_{\mu\nu} = \pi_{\mu\nu\alpha\beta}\sigma_{\alpha\beta}. \tag{28}$$

But, as one can see easily, this $\sigma_{\mu\nu}$ does not satisfy (21). So, the self duality does not hold for $F_{\mu\nu}$. It is not easy to guess a matrix that can satisfy a self duality and the YM equation of motion at the same time.

In summary, there is two ways, either to modify the 't Hooft like solution or to change the self duality formulation. We prefer to choose the second way. We have a Cliff(0,7) structure in our formulation so

$$E_i E_j = \rho_{ijk} E_k \Longleftrightarrow E_i E_j = constant \ \ \epsilon_{ijlmnpk} E_k, \tag{29}$$

and in 8 dimensions, we have the standard $\epsilon_{\alpha\beta\gamma\delta\zeta\xi\mu\nu}$ Levi-Cevita tensor. Having such tools, in principle, may be useful to study a new type of gravitational instanton

$$R_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta\zeta\xi\mu\nu} R_{\zeta\xi\mu\nu}. \tag{30}$$

where $R$ is our standard Reimanian tensor but may be formed from a torsionful connection. Such point is under investigation.

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